

## INVENTORY MODEL WITH FUZZY LEAD-TIME AND DYNAMIC DEMAND OVER FINITE TIME HORIZON USING INTERACTIVE FUZZY METHOD

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### ABSTRACT

The real-world inventory control problems are normally imprecisely defined and human interventions are often required in solving these decision-making problems. In this paper, a realistic inventory problem with infinite rate of replenishment over a prescribed finite time horizon is developed considering time dependent demand, which increases with time and imprecise lead time. Shortages are allowed and backlogged partially. The imprecise lead-time is here assumed to be represented by linear membership function. The imprecise parameter is first transformed to corresponding interval numbers and then following the interval mathematics, the objective function for average cost is changed to respective multi objective functions. These functions are minimized and solved for a pareto optimum solution by interactive fuzzy decision making procedure using a logic structure. The impreciseness of lead-time and man-machine interaction lead to a multiple logical decision process. This leads to man-machine interaction for optimum and appropriate decision acceptable to the decision maker's firm / company. The model is illustrated numerically and the results are presented in algorithmic and tabular forms.

**KEYWORDS:** Fuzzy Lead-Time, Interval Number, Crisp Inventory Model, Interactive Fuzzy Decision Making Method, Pareto Optimal Solution.

### INTRODUCTION

Since the development of EOQ model by Harris [1], lot of research works have been carried out in inventory control system. In the existing literature, inventory models are generally developed under the assumption of constant or stochastic lead-time. A number of research papers have already been published in this direction (cf. Das [2], and Foote et.al [3] etc). Recently, Kalpakam and Swapan [4] studied a perishable inventory model with stochastic lead-time. But in real life situations, the lead-time is normally vague and imprecise i.e. uncertain in non-stochastic sense. It will be more realistic to consider the lead-time as fuzzy in nature.

Normally, duration of seasonal products is constant and these are available in the market every year during a fixed interval of time. Hence the time period for the business of seasonal goods is finite. Several

researchers (Hariaga and Benkharouf [5], Chakraborty and Chaudhari [6], Bhuina and Maiti [7] etc.) have developed this type of inventory models also.

In multi-objective mathematical programming problems, a decision maker is required to maximize/minimize two or more objectives simultaneously over a given set of possible situations. A number of methods, assigning priorities to the objectives, setting aspiration level for the objectives, etc. exists for finding compromise solutions of multi-criteria decision making problems. Recently, Roy and Maiti [8] developed a multi-objective inventory model for deteriorating items with stock dependent demand under two restrictions in fuzzy environment. They solved the problem with infinite time horizon not considering shortages.

In inventory system, shortage may occur due to different causes, viz. delayed supply /production, transportation problem, sudden increase of demand, artificial crisis etc. Though shortages bring loss of goodwill, still allowing shortage is one of the managerial decisions for business.

In a fuzzy programming problem, the parameters are normally defined by fuzzy numbers. The fuzzy numbers describe the imprecise coefficients of a fuzzy model. These imprecise coefficients may then approximate to a crisp set of interval numbers. Grzegorzewski [9] suggested a method to substitute a fuzzy set by a crisp one. Chanas and Kutcha [10] defined a transportation problem with fuzzy cost coefficients and developed an algorithm to solve the problem replacing the fuzzy parameters by crisp interval numbers. In a fuzzy interactive linear / non-linear multi-objective decision making problem, DM plays an important role. He has every right to choose the suitable membership functions to achieve the optimum goal. In this way, an interaction is established with the DM. Sakawa [11,12] proposed a new technique to solve such type of problems.

This paper develops an inventory problem with time dependent demand rate for a prescribed finite time horizon allowing imprecise lead-time. The lead-time is represented by a fuzzy number. The fuzzy number is expressed with the help of a linear membership function and then converted to appropriate interval numbers following Grzegorzewski [9]. Here, shortages are allowed but the item is assumed to be so costly that there is a restriction on the shortage levels. There may be six models (model-1,2,3,4,5,6) depending upon the nature of first and last cycles. For each model, there will be different scenarios (total 6 scenarios) depending upon the time of placement of order for the next lot in the first cycle. Again, for each scenario, there is a number of cases (total  $6 \times 4N$   $12 \times 4N - 1$  models for the system) depending upon the placement time for the next orders during successive time cycles. For each problem using the concept of interval arithmetic, we have constructed an equivalent multi-objective deterministic problem corresponding to the original problem with interval co-efficient. This equivalent problem has been solved using interactive fuzzy decision making procedure and allowing man-machine interaction to choose different type of membership functions for the multi-objectives. Finally, some numerical examples are used to illustrate the models how it work.

### FORMULATION OF THE MULTI-OBJECTIVE PROBLEM

We define a general non-linear objective function with coefficients of the decision variables as interval valued numbers

$$\text{Minimize } Z(x) = \frac{\sum_{i=1}^k [a_{L_i}, a_{R_i}] \prod_{j=1}^n x_j^{r_j}}{\sum_{i=1}^l [b_{L_i}, b_{R_i}] \prod_{j=1}^n x_j^{q_j}} \quad (1)$$

subject to  $x_j > 0, j = 1, 2, \dots, n$  and  $x \in S \subset \mathbb{R}^n$

where  $S$  is the feasible region of  $x, 0 \leq a_{L_i} \leq a_{R_i}, 0 \leq b_{L_i} \leq b_{R_i}$  and  $r_i, q_j$  are positive numbers.

Now, we exhibit the formulation of the original problem (1) as a multi-objective non-linear problem. Since the objective function  $Z(x)$  is an interval, it is natural that the solution set of (1) should be defined by preference relations between intervals.

Now from equation (1), following the interval arithmetic's (cf. Moore [13], Inuiguchi and Kume[14]) we have

$$\begin{aligned} Z(x) &= \frac{\sum_{i=1}^k [a_{C_i} - a_{W_i}, a_{C_i} + a_{W_i}] \prod_{j=1}^n x_j^{r_j}}{\sum_{i=1}^l [b_{C_i} - b_{W_i}, b_{C_i} + b_{W_i}] \prod_{j=1}^n x_j^{q_j}} \quad \text{where } a_{C_i} = \frac{a_{L_i} + a_{R_i}}{2}, \quad a_{W_i} = \frac{a_{R_i} - a_{L_i}}{2}, \\ &= \left[ \frac{\sum_{i=1}^k \left( a_{C_i} \prod_{j=1}^n x_j^{r_j} - a_{W_i} \prod_{j=1}^n |x_j|^{r_j} \right)}{\sum_{i=1}^l \left( b_{C_i} \prod_{j=1}^n x_j^{q_j} + b_{W_i} \prod_{j=1}^n |x_j|^{q_j} \right)}, \frac{\sum_{i=1}^k \left( a_{C_i} \prod_{j=1}^n x_j^{r_j} + a_{W_i} \prod_{j=1}^n |x_j|^{r_j} \right)}{\sum_{i=1}^l \left( b_{C_i} \prod_{j=1}^n x_j^{q_j} - b_{W_i} \prod_{j=1}^n |x_j|^{q_j} \right)} \right] \quad (2) \end{aligned}$$

The right limit  $Z_R(x)$  of the interval objective function  $Z(x)$  may be elicited as

$$z_R(x) = \frac{\sum_{i=1}^k \left( a_{C_i} \prod_{j=1}^n x_j^{r_j} + a_{W_i} \prod_{j=1}^n |x_j|^{r_j} \right)}{\sum_{i=1}^l \left( b_{C_i} \prod_{j=1}^n x_j^{q_j} - b_{W_i} \prod_{j=1}^n |x_j|^{q_j} \right)} = \frac{\sum_{i=1}^k a_{R_i} \prod_{j=1}^n x_j^{r_j}}{\sum_{i=1}^l b_{L_i} \prod_{j=1}^n x_j^{q_j}} \quad (3)$$

Similarly the left limit  $Z_L(x)$  may be written as

$$Z_L(x) = \frac{\sum_{i=1}^k \left( a_{C_i} \prod_{j=1}^n x_j^{r_j} - a_{W_i} \prod_{j=1}^n |x_j|^{r_j} \right)}{\sum_{i=1}^l \left( b_{C_i} \prod_{j=1}^n x_j^{q_j} + b_{W_i} \prod_{j=1}^n |x_j|^{q_j} \right)} = \frac{\sum_{i=1}^k a_{L_i} \prod_{j=1}^n x_j^{r_j}}{\sum_{i=1}^l b_{R_i} \prod_{j=1}^n x_j^{q_j}} \quad (4)$$

The center of the objective function  $Z_C(x)$  can be written as

$$Z_C(x) = \frac{1}{2} [Z_R(x) + Z_L(x)] \quad (5)$$

Thus the problem (1) is transformed into

$$\text{Minimize } \{ Z_L, Z_R, Z_C \} \quad (6)$$

subject to the non-negativity constraints of (1), where  $Z_R$ ,  $Z_L$  and  $Z_C$  are defined by the equations (3), (4) and (5) respectively.

### The Nearest Interval Approximation

Here we want to approximate a fuzzy number by a crisp interval. Suppose  $\tilde{A}$  and  $\tilde{B}$  are two

fuzzy numbers with  $\alpha$ -cuts i.e.  $[A_L(\alpha), A_R(\alpha)]$  and  $[B_L(\alpha), B_R(\alpha)]$  respectively. Then the distance between  $\tilde{A}$  and  $\tilde{B}$  is

$$d(\tilde{A}, \tilde{B}) = \sqrt{\int_0^1 (A_L(\alpha) - B_L(\alpha))^2 d\alpha + \int_0^1 (A_R(\alpha) - B_R(\alpha))^2 d\alpha}.$$

Given  $\tilde{A}$  is a fuzzy number, we have to find a closed interval  $C_d(\tilde{A})$  which is the nearest to  $\tilde{A}$  with respect to metric  $d$ . We can do it since each interval is also a fuzzy number with constant  $\alpha$ -cut for all  $\alpha \in [0, 1]$ . Hence  $(C_d(\tilde{A}))_{\alpha} = [C_L, C_R]$ .

Now we have to minimize

$$d(\tilde{A}, C_d(\tilde{A})) = \sqrt{\int_0^1 (A_L(\alpha) - C_L)^2 d\alpha + \int_0^1 (A_R(\alpha) - C_R)^2 d\alpha} \quad (7)$$

with respect to  $C_L$  and  $C_R$ .

In order to minimize  $d(\tilde{A}, C_d(\tilde{A}))$ , it is sufficient to minimize the function  $D(C_L, C_R) (= d^2(\tilde{A}, C_d(\tilde{A})))$ . The first partial derivatives are

$$\frac{\delta D(C_L, C_R)}{\delta C_L} = -2 \int_0^1 A_L(\alpha) d\alpha + 2C_L$$

and

$$\frac{\delta D(C_L, C_R)}{\delta C_R} = -2 \int_0^1 A_R(\alpha) d\alpha + 2C_R$$

Solving,  $\frac{\delta D(C_L, C_R)}{\delta C_L} = 0$  and  $\frac{\delta D(C_L, C_R)}{\delta C_R} = 0$  we get

$$C_L^* = \int_0^1 A_L(\alpha) d\alpha \quad \text{and} \quad C_R^* = \int_0^1 A_R(\alpha) d\alpha. \quad (8)$$

Again since  $\frac{\delta D^2(C_L^*, C_R^*)}{\delta C_L^2} = 2 > 0$ ,  $\frac{\delta D^2(C_L^*, C_R^*)}{\delta C_R^2} = 2 > 0$  and

$$H(C_L^*, C_R^*) = \frac{\delta D^2(C_L^*, C_R^*)}{\delta C_L^2} \cdot \frac{\delta D^2(C_L^*, C_R^*)}{\delta C_R^2} - \left( \frac{\delta D^2(C_L^*, C_R^*)}{\delta C_L \cdot \delta C_R} \right)^2 = 4 > 0$$

so  $D(C_L, C_R)$  i.e.  $d(\tilde{A}, C_d(\tilde{A}))$  is global minimum.

Therefore the interval  $C_d(\tilde{A}) = \left[ \int_0^1 A_L(\alpha) d\alpha, \int_0^1 A_R(\alpha) d\alpha \right]$  (9)

is the nearest interval approximation of fuzzy number  $A$  with respect to the metric  $d$ .

Let  $\tilde{A} = (a_1, a_2, a_3)$  be a fuzzy number. The  $\alpha$ -level interval of  $\tilde{A}$  is defined as  $(\tilde{A})_\alpha = [A_L(\alpha), A_R(\alpha)]$ .

When  $\tilde{A}$  is a TFN, then  $A_L(\alpha) = a_1 + \alpha(a_2 - a_1)$  and  $A_R(\alpha) = a_3 + \alpha(a_3 - a_2)$ .

By the nearest interval approximation method the lower limit of the interval is

$$\begin{aligned} C_L &= \int_0^1 A_L(\alpha) d\alpha \\ &= \int_0^1 [a_1 + \alpha(a_2 - a_1)] d\alpha = \frac{1}{2} (a_2 + a_1) \end{aligned} \quad (10)$$

And the upper limit of the interval is

$$\begin{aligned} C_R &= \int_0^1 A_R(\alpha) d\alpha \\ &= \int_0^1 [a_3 - \alpha(a_3 - a_2)] d\alpha = \frac{1}{2} (a_2 + a_3) \end{aligned} \quad (11)$$

Therefore the interval number considering  $\tilde{A}$  as a TFN is  $[(a_1+a_2)/ 2, (a_2+a_3)/ 2]$ .

Similarly, when  $A$  is a PFN then  $A_L(\alpha) = a_2 - (a_2 - a_1) \vee (1 - \alpha)$  and  $A_R(\alpha) = a_2 + (a_3 - a_2) \vee (1 - \alpha)$ .

Following the same way stated above, the interval number is  $[(2a_1+a_2)/ 3, (a_2+2a_3)/ 3]$ .

**PROBLEM FORMULATION**

The inventory model with imprecise lead-time and dynamic demand is developed for the prescribed finite time horizon under the following assumptions and notations.

**ASSUMPTIONS AND NOTATIONS**

- (i) Rate of replenishment is infinite.
- (ii) Shortages are allowed but backlogged partially.
- (iii) The entire lot is delivered in one batch.
- (iv) Inventory system involves only one item and one stocking point.
- (v) There is no quantity discount.
- (vi)  $f(t) = a.t^2 + b.t + c$ ,  $a, b, c > 0$ , be the deterministic quadratic demand per unit time, which increases with time.
- (vii)  $C_1$  = The inventory carrying cost per unit per unit time for each cycle.
- (viii)  $C_2$  = Shortage cost per unit per unit time for each cycle.
- (ix)  $C_3$  = The replenishment (ordering) cost per order.
- (x)  $t_1$  = Length of the time when new order is placed.
- (xi)  $t_2$  = Length of the time when inventory reaches zero. .
- (xii)  $t_3$  = Length of each cycle.
- (xiii)  $H$  = Prescribed time horizon.
- (xiv)  $N$  = Total number of replenishments to be made during the prescribed time horizon  $H$ .
- (xv)  $L$  = Lead-time, which is a fuzzy number i.e.  $L = (a_1, a_2, a_3) \equiv [L_1, L_2]$ ,  $(0 < L_1 < L_2)$ .

**For j-th cycle (j=1,2,3,...,N+1) :**

- (xvi)  $C_j$  = purchasing cost per unit quantity and is dependent upon the lead time  $L$  such that  $C_j = CP + CP' / L$ ,  $CP' > 0$ .
- (xvii)  $q_j(t)$  = inventory level at time  $t$ .
- (xviii)  $Q_j$  = inventory level.
- (xix)  $S_j$  = on hand inventory when the new order is placed for the next cycle.
- (xx)  $R_j$  = shortage level.
- (xxi)  $T_j = (j - 1) t_3$ .

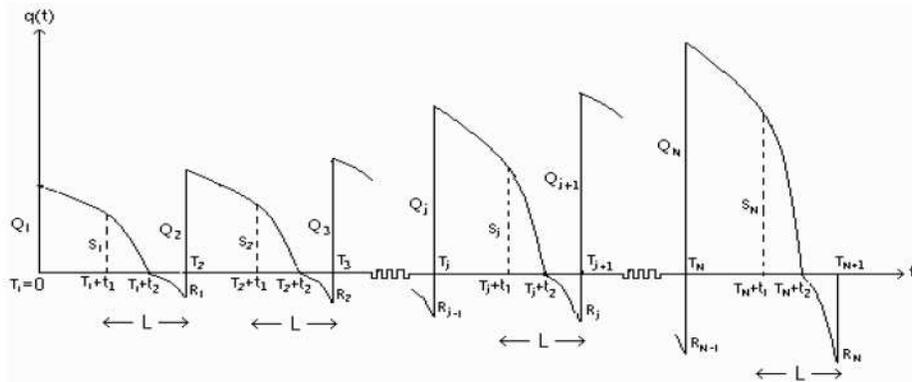
**MATHEMATICAL MODELS**

In an inventory situation, for a fixed prescribed time horizon with number of cycles, a retailer or manufacturer may have different options during starting and closing of his/her business. At the beginning of the cycle, one may start with (i) some replenishment/procurement ( model-1,2,3 ) or (ii) allowing the shortages for the items which are later partially backlogged ( model-4,5,6 ) . Similarly, towards the end of the last cycle, one can wound up the business (i) allowing shortages and later partially backlogged only (model-1, 4) or (ii) allowing shortages but later do not backlogged them ( model-3,6 ) or (iii) with the exhaust of the stock, not allowing further shortages (model-2,5). Here different inventory models (model-1, 2,3,4,5,6) have been discussed combining the above mentioned situations. Let there be (N+1) cycles during the fixed time horizon, H.

**Model-1**

In this model, the shortages are allowed at the end of each cycle. The j-th cycle (for j=1,2,...,N+1) starts with inventory  $Q_j$  units at  $t=T_j$  and shortages are allowed to be accumulated upto  $R_j$  units at  $t=T_{j+1}$ . The procurement of  $(Q_j+R_j-1)$  units first satisfies the shortages at  $t=T_j$  and then the rest of the procurement is kept in store to meet the demand during  $[T_j, T_{j+1}]$  , for  $j=2,3,...,N+1$ .

In the last cycle (i.e., in (N+1) th cycle), only the shortage units are replenished. Here,  $H= (N+1)t_3$ .

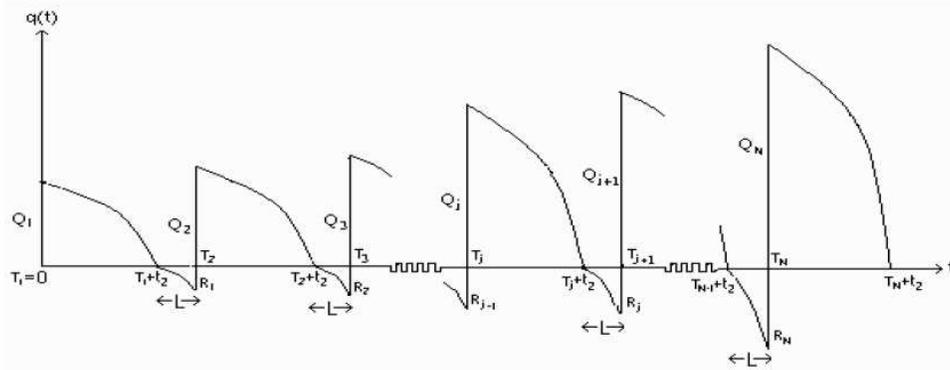


**Fig-1: Pictorial representation of Model-1 (one situation)**

**Model-2**

In this model, the shortages are allowed at the end of each cycle except the last one. The j-th cycle (for j=1,2,...,N) starts with inventory  $Q_j$  units at  $t=T_j$  and shortages are allowed to be accumulated upto  $R_j$  units at  $t=T_{j+1}$ . The procurement of  $(Q_j+R_j-1)$  units (for  $j=2,3,...,N+1$ ) first satisfies the shortages at  $t=T_j$  and then the rest of the procurement is kept in store to meet the demand during  $[T_j, T_{j+1}]$ .

In the last cycle, shortages are not allowed. Here,  $H=Nt_3+t_2$ .

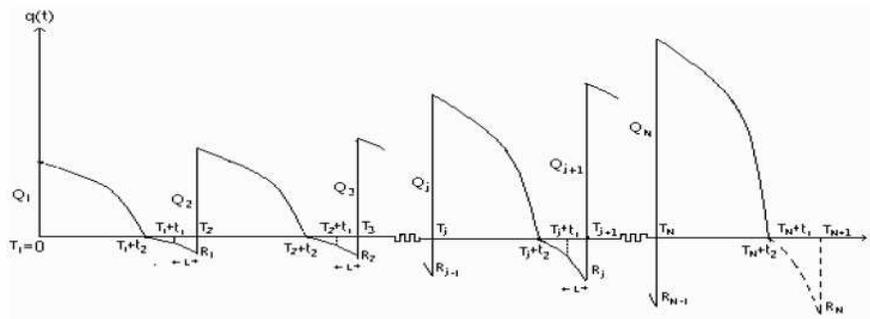


**Fig-2: Pictorial representation of Model-2 (one situation)**

**Model-3**

In this model, the shortages are allowed at the end of each cycle. The  $j$ -th cycle (for  $j=1,2,\dots,N+1$ ) starts with inventory  $Q_j$  units at  $t=T_j$  and shortages are allowed to be accumulated upto  $R_j$  units at  $t=T_{j+1}$ . The procurement of  $(Q_j+R_j-1)$  units first satisfies the shortages at  $t=T_j$  and then the rest of the procurement is kept in store to meet the demand during  $[T_j, T_{j+1}+t_2]$ , for  $j=2,3,\dots,N+1$ .

In the last cycle (i.e., in  $(N+1)$  th cycle), the shortage units are not backlogged. Here,  $H= (N+1) t_3$ .



**Fig-3: Pictorial representation of Model-3 (one situation)**

Four different scenarios may arise depending upon the reorder point for the second cycle of above three models:

**Scenario-I:** The order will be placed at  $T_1$  when on hand inventory becomes  $Q_1$  (i.e., order will be placed at the time of replenishment).

**Scenario-II:** The order will be placed at  $T_1+ t_1 (< T_1+ t_2)$  when on hand inventory becomes  $S_1 (< Q_1)$ .

**Scenario-III:** The order will be placed at  $T_1+ t_2 (= T_1+ t_1)$  when inventory level reaches zero.

**Scenario-IV:** The order will be placed at  $T_1 + t_1 (> T_1 + t_2)$  after starting shortage and when shortage level is  $S_1 (< R_1)$ .

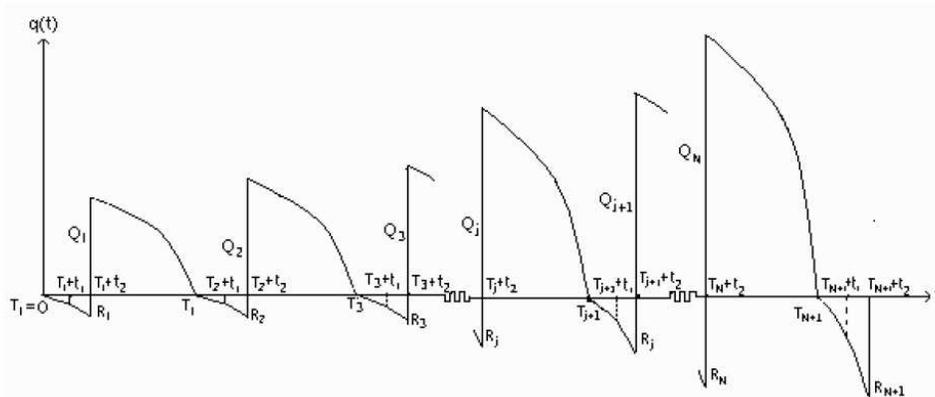
In each scenario's, the order for the next cycle can be placed at the time of receiving the present consignment or at a time between the receipt of the present order and occurrence of the shortages or at the time when shortages begin or at a time during the shortage period. Let, the above mentioned four situations be denoted by A, B, C, D respectively. Hence, for each scenario of the above models, combining all the possibilities of placing the order for the successive cycles, there will be  $4N+1$  cases for model-1 and  $4N$  cases for model-2 and 3. Here, for model-I,3 and model-2, the inventory control system for only one case is presented since the other cases can be easily derived following the illustrated methodology.

**Case-I:** In this case, the orders for every consecutive cycle are placed at a time between the receipt of the present order and occurrence of the shortages, and the cycles start with inventory.

**Model-4**

In this model, the shortages occur at the beginning and then stock is built up at each cycle after backlogging the shortages except the last one. The  $j$ -th cycle (for  $j=1,2,\dots,N$ ) starts with zero inventory and shortages are allowed to be accumulated upto  $R_j$  units. The procurement of  $(Q_j+R_j)$  units first satisfies the shortages and then the rest of the procurement is kept in store to meet the demand during no shortage period.

In the last cycle, only shortages units are replenished. Here,  $H= N.t_3+ L$ .

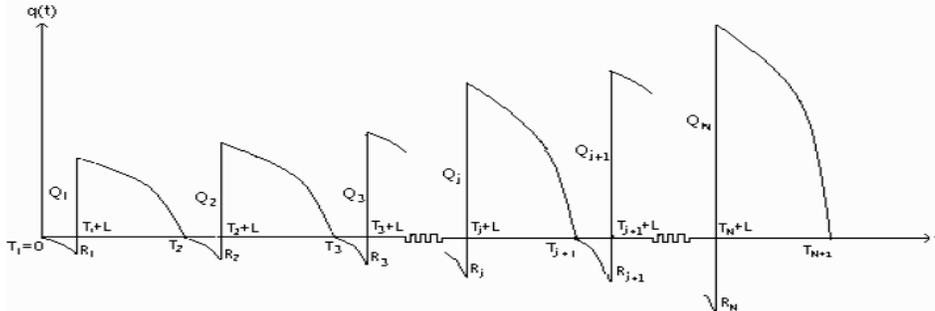


**Fig-4: Pictorial representation of Model-4 (one situation)**

**Model-5**

In this model, the shortages occur at the beginning and then stock is built up at each cycle after backlogging the shortages. The  $j$ -th cycle (for  $j=1,2,\dots,N$ ) starts with zero inventory and shortages are allowed to be accumulated upto  $R_j$  units. The procurement of  $(Q_j+R_j)$  units first satisfies the shortages

and then the rest of the procurement is kept in store to meet the demand during no shortage period. Here,  $H = Nt_3$ .

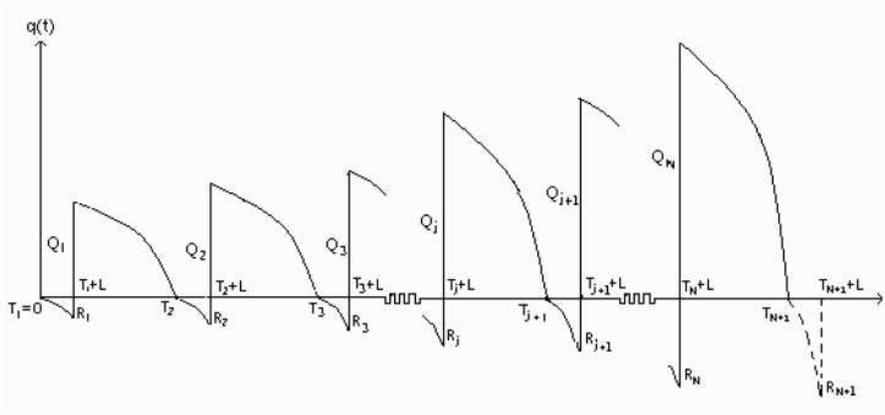


**Fig-5: Pictorial representation of Model-5 (one situation)**

**Model-6**

In this model, the shortages occur at the beginning and then stock is built up at each cycle after backlogging the shortages except the last one. The  $j$ -th cycle (for  $j=1,2,\dots,N$ ) starts with zero inventory and shortages are allowed to be accumulated upto  $R_j$  units. The procurement of  $(Q_j+R_j)$  units first satisfies the shortages and then the rest of the procurement is kept in store to meet the demand during no shortage period.

In the last cycle, the shortages units are not backlogged. Here,  $H = N.t_3 + L$ .



**Fig-6: Pictorial representation of Model-6 (one situation )**

Two different scenarios may arise depending upon the recorder point for the first cycle of the above three models:

**Scenario-I:** The order will be placed at  $T_1(=0)$ .

**Scenario-II:** The order will be placed at  $T_1 + t_1$  ( $t_1 < t_2$ ) during shortage period and when shortage level is  $S_1$ .

Depending upon the placing of order for the next cycle in each scenarios, there will be four situations A, B, C, D (which are mentioned earlier) also occurs. Hence, for each scenario of the above models, combining all the possibilities of placing the order for the successive cycles, there will be  $2 \times 4N^{-1}$  cases for each model-4,6 and  $2 \times 4N$  cases for model-4.

Here, for model-4,5 and model-6, the inventory control system for only one case is presented since the other cases can be easily derived following the illustrated methodology.

**Case-II:** In this case, the order for the first consignment is placed at the beginning of the system which starts with shortages and the orders for other consecutive cycles are placed during the shortage period.

**Formulation of case-I ( for model-1 )**

In this scenario, the inventory level  $q_j(t)$  at time  $t$  ( $T_j \leq t \leq T_{j+1}$  ,  $j=1,2,3,\dots,N+1$ .) satisfies the following differential equations:

$$\frac{dq_j(t)}{dt} = \begin{cases} -f(t) , & T_j \leq t \leq T_j + t_2 \\ -\delta f(T_j + t_2), & T_j + t_2 \leq t \leq T_{j+1} \end{cases} \quad (12)$$

with the boundary conditions

$$\begin{aligned} q_j(t) &= Q_j, & t &= T_j \\ &= 0, & t &= T_j + t_2 \\ &= -R_j, & t &= T_{j+1} \end{aligned} \quad (13)$$

and  $q_j(t) = S_j$  at  $t = T_j + t_1$

The solutions of the differential equations (12) with the help of (16) are

$$q_j(t) = \begin{cases} \frac{a}{3} \left\{ (T_j + t_2)^3 - t^3 \right\} + \frac{b}{2} \left\{ (T_j + t_2)^2 - t^2 \right\} + \\ c \left\{ (T_j + t_2) - t \right\}, & T_j \leq t \leq T_j + t_2 \\ \delta (T_j + t_2 - t) f(T_j + t_2) , & T_j + t_2 \leq t \leq T_{j+1} \end{cases} \quad (14)$$

Using the condition (16) and (17), we get from (18)

$$Q_j = at_2 T_j^2 + (at_2^2 + bt_2) T_j + \left( \frac{a}{3} t_2^3 + \frac{b}{2} t_2^2 + ct_2 \right) \quad (15)$$

and  $R_j = \delta (t_3 - t_2) f(T_j + t_2)$  (16)

The total inventory carrying cost of the system is given by

$$\begin{aligned}
C_{Hj} &= C_1 \int_{T_j}^{T_j+t_2} q_j(t) dt \\
&= C_1 \left\{ \frac{a}{2} t_2^2 T_j^2 + \left( \frac{2a}{3} t_2^3 + \frac{b}{2} t_2^2 \right) T_j + \left( \frac{a}{4} t_2^4 + \frac{b}{3} t_2^3 + \frac{c}{2} t_2^2 \right) \right\}
\end{aligned} \tag{17}$$

The total shortage cost of the system is given by

$$\begin{aligned}
C_{Sj} &= -C_2 \int_{T_j+t_2}^{T_{j+1}} q(t) dt \\
&= \frac{1}{2} \delta C_2 f(T_j + t_2) (t_3 - t_2)^2
\end{aligned} \tag{18}$$

Total cost over the time horizon is given by

$$\begin{aligned}
F(t_1, t_2) &= \sum_{j=1}^{N+1} C_{Hj} + \sum_{j=1}^{N+1} C_{Sj} + \left( C + \frac{C'}{L} \right) \sum_{j=1}^{N+1} Q_j + \left( C + \frac{C'}{L} \right) \sum_{j=1}^{N+1} R_j + \sum_{j=1}^{N+1} C_{3j} \\
&= [F_L, F_R]
\end{aligned} \tag{19}$$

(For formulation of  $F_L, F_R$ , see **Appendix-1**)

Using the equation (6), our problem given by (19) may be rewritten as

$$\begin{aligned}
&\text{Minimize } \{ F_L, F_R, F_C \} \\
&\text{where } F_C = \frac{1}{2} (F_L + F_R). \\
&\text{Obviously } 0 \leq S_j \leq Q_j \text{ for } j=1,2,\dots,N+1.
\end{aligned} \tag{20}$$

### Interactive Approach For Solution:

Above interval problem is now reduced to a multi objective non-linear programming problem as

$$\text{Minimize} \{ F_L(N, t_2), F_R(N, t_2), F_C(N, t_2) \} \tag{21}$$

Now interactive approaches be used by considering the imprecise nature of the DM's judgement, which is natural to assume that the DM may have fuzzy or imprecise goals for each of the objective functions

To derive the membership functions  $\mu_{F_L}, \mu_{F_R}, \mu_{F_C}$  each of the objective function  $F_L, F_R, F_C$  from DM's view point we first calculate individual minimum ( i.e.  $F_L^{\min}, F_R^{\min}, F_C^{\min}$  ) and individual maximum ( i.e.  $F_L^{\max}, F_R^{\max}, F_C^{\max}$  ) by a non-linear optimization technique.

With the help of individual minimum and maximum, the DM can select his membership functions from among two types of membership functions i.e. from

- (i) linear membership function
- (ii) quadratic membership function

The membership functions  $\mu_{F_L}$ ,  $\mu_{F_R}$  and  $\mu_{F_C}$  for each of the objective functions  $F_L$ ,  $F_R$  and  $F_C$

$$\text{may be written as } \mu_{F_K} = \begin{cases} 1 & \text{if } F_K \leq F_K^1 \\ d_K & \text{if } F_K^1 \leq F_K \leq F_K^0 \\ 0 & \text{if } F_K \geq F_K^0 \end{cases} \quad (22)$$

where  $F_K^1$  and  $F_K^0$  are to be chosen such that  $F_K^{\min} \leq F_K^1 \leq F_K^0 \leq F_K^{\max}$  and  $d_K$  is a strictly monotonic decreasing continuous function of  $F_K$  which may be linear or non-linear.

**LINEAR MEMBERSHIP FUNCTION (TYPE-I)**

For each objective function, the corresponding linear membership functions are as follows:

$$\mu_{F_K} = \begin{cases} 1 & \text{if } F_K \leq F_K^1 \\ 1 - \frac{F_K - F_K^1}{P_K} & \text{if } F_K^1 \leq F_K \leq F_K^0 \\ 0 & \text{if } F_K \geq F_K^0 \end{cases} \quad (23)$$

where  $F_K^1$  and  $F_K^0$  are to be chosen such that  $F_K^{\min} \leq F_K^1 \leq F_K^0 \leq F_K^{\max}$  and  $P_K = F_K^0 - F_K^1$  is the tolerance of k-th objective function  $F_K$ .

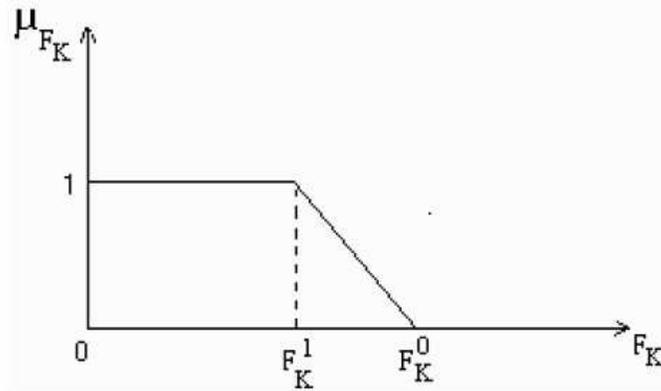


Fig. 7 Pictorial representation of linear  $\mu_{F_K}$

**QUADRATIC MEMBERSHIP FUNCTION (TYPE-II)**

For each of the objective functions the corresponding quadratic membership functions are

$$\mu_{F_K} = \begin{cases} 1 & \text{if } F_K \leq F_K^1 \\ 1 - \left( \frac{F_K - F_K^1}{P_K} \right)^2 & \text{if } F_K^1 \leq F_K \leq F_K^0 \\ 0 & \text{if } F_K \geq F_K^0 \end{cases} \quad (24)$$

where  $F_K^1$  and  $F_K^0$  are to be chosen such that  $F_K^{\min} \leq F_K^1 \leq F_K^0 \leq F_K^{\max}$  and  $P_K = F_K^0 - F_K^1$  is the tolerance of k-th objective function  $F_K$ .

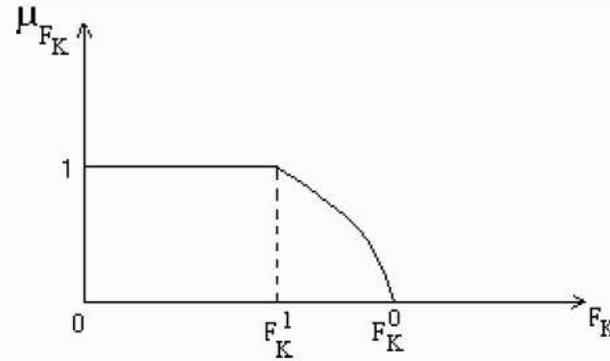


Fig. 8 Pictorial representation of quadratic  $\mu_{F_K}$

### FUZZY DECISION MAKING METHOD

After determining the different linear / non-linear membership functions for each of the objective functions, Bellman and Zedah [15] and following Zimmermann [16] the given problem (19) can be formulated as

$$\begin{aligned} & \text{Max } \lambda \\ & \text{subject to } \lambda \leq \mu_{F_L}, \lambda \leq \mu_{F_R}, \lambda \leq \mu_{F_C}, \\ & 0 \leq \lambda \leq 1. \end{aligned} \quad (25)$$

With the help of two different type of membership functions given by (23) and (24), above problem can be restated for a particular choice of DM as

$$\begin{aligned} & \text{Max } \lambda \\ & \text{subject to } \lambda \leq 1 - \frac{F_L - F_L^1}{P_L}, \quad \text{if first objective} \in \text{Type - I.} \\ & \lambda \leq 1 - \left( \frac{F_R - F_R^1}{P_R} \right)^2, \quad \text{if second objective} \in \text{Type - II.} \\ & \lambda \leq 1 - \frac{F_C - F_C^1}{P_C}, \quad \text{if third objective} \in \text{Type - I.} \\ & 0 \leq \lambda \leq 1. \end{aligned} \quad (26)$$

Here DM selects the above membership functions for the corresponding objective functions. Then the above problem can be solved by non-linear optimization technique and optimal solution of  $\lambda$ , says  $\lambda^*$  is obtained.

Now the DM selects his most important objective functions from among the objective functions FL, FR and FC. Here FR is selected as DM would like to minimize his / her worst case. Then the problem becomes (for  $\lambda = \lambda^*$ )

$$\begin{aligned} & \text{Min } F_R \\ & \text{subject to } F_L \leq m_L, \quad F_R \leq m_R, \quad F_C \leq m_C, \\ & \quad \quad \quad 0 \leq \lambda \leq 1 \end{aligned} \quad (27)$$

$$\begin{aligned} \text{where } m_L &= F_L^1 + P_L(1 - \lambda^*), & \text{if the first objective } \in \text{Type - I.} \\ m_R &= F_R^1 + P_R \sqrt{1 - \lambda^*}, & \text{if the second objective } \in \text{Type - II.} \\ m_C &= F_C^1 + P_C(1 - \lambda^*), & \text{if the third objective } \in \text{Type - I.} \end{aligned}$$

### PARETO OPTIMAL SOLUTION

Here Pareto optimality test is performed according to Sakawa [13]. Let the decision vector  $t_1^*$ ,  $t_2^*$  and the optimum values,  $F_L^* = F_L(t_1^*, t_2^*)$ ,  $F_R^* = F_R(t_1^*, t_2^*)$  and  $F_C^* = F_C(t_1^*, t_2^*)$  are obtained from (27). Then solve the problem:

$$\begin{aligned} & \text{Minimize } V = (F_L + F_R + F_C) \\ & \text{subject to } F_L \leq F_L^*, \quad F_R \leq F_R^*, \quad F_C \leq F_C^*, \\ & \quad \quad \quad 0 \leq \lambda \leq 1. \end{aligned} \quad (28)$$

Using a non-linear optimization technique, the optimal solution of (28), say,  $\overline{t_1}, \overline{t_2}, \overline{Q}, \overline{Q_1}, \overline{F_L}, \overline{F_R}$  and  $\overline{F_C}$  is strong Pareto optimal solutions provided V is very small, otherwise it is weak Pareto optimal solution.

### Numerical Example

To illustrate the proposed inventory models, following input data are considered.

**Input Data:**  $C_1=0.4$ ,  $C_2=6.25$ ,  $C_3=420$ ,  $\delta=0.93$ ,  $a=0.25$ ,  $b=1$ ,  $c=100$ ,  $C_p=15$ ,  $C'=0.15$ ,  $H_1=12$ ,  $H_2=14$ ,  $a_1=0.45$ ,  $a_2=0.65$ ,  $a_3=0.85$  in proper units.

### WHEN FUZZY PARAMETER IS TFN

Considering the above fuzzy parameter  $L_{\sim}$  as triangular fuzzy number (TFN), the nearest interval approximations according to Grzegorzewski [9] is  $L_{\sim} \equiv [ .55, .75 ]$ .

Following (23) and (25), the problem (21) is solved and the results are presented in the following tables:

**Table-1. Individual Minimum and Maximum of Objective Functions**

Objective functions	Minimum			Maximum		
	Model-I	Model -2	Model -3	Model -I	Model -2	Model -3
$F_L$	24941.63	25346.60	24191.51	24973.73	25459.94	24292.55
$F_R$	29054.57	29642.79	28401.09	29087.24	29758.58	28504.16
$F_C$	27006.20	27523.34	26321.81	27014.44	27552.59	26347.83

Table-2  
(Input Data for  $F_K^1, F_K^0$ )

Model	$F_L^1$	$F_L^0$	$F_R^1$	$F_R^0$	$F_C^1$	$F_C^0$
Model-I	24941.63	24970.73	29054.57	29080.24	27006.20	27014.00
Model-2	25346.60	25440.94	29642.79	29750.58	27523.34	27550.59
Model-3	24191.51	24280.55	28401.09	28500.16	26321.81	26347.00

**Solution with Fuzzy Decision Making Method**

DO YOU WANT LIST OF MEMEBERSHIP FUNCTIONS ?

= YES

LIST OF MEMBERSHIP FUNCTIONS

(1) LINEAR

(2) QUADRATIC

INPUT MEMBERSHIP FUNCTION TYPE FOR FIRST OBJECTIVE:

= 1

INPUT MEMBERSHIP FUNCTION TYPE FOR SECOND OBJECTIVE:

= 2

INPUT MEMBERSHIP FUNCTION TYPE FOR THIRD OBJECTIVE:

= 1

Let, at the beginning, analysis is performed to find optimum  $\square$  with the membership function

$F_L, F_C$  as linear (Type-I) and  $F_R$  as Quadratic (Type-II). The optimum value of  $\square$  is presented in

**Table: 3.λ- MAXIMUM CALCULATION (following (26))**

Maximum $\lambda$	Model-I	Model-2	Model-3
$\lambda^*$	0.8101376	0.8278420	0.8378888

With this value of  $\lambda^*$  following(27), the objective function  $F_R$  is optimized and the optimum results are:

Table-4

(optimal results when choosing most important objective function as  $F_R$  )

Model	N	$t_1^*$	$t_2^*$	$[ F_L^*, F_R^* ]$	$F_C^*$
Model-I	9	0.65	0.9770460	[ 24947.15 , 29065.76 ]	27006.46
Model-2	10	0.5579924	0.9200764	[ 25362.84 , 29687.51 ]	27525.18
Model-3	9	0.65	0.7248860	[ 24205.95 , 28440.97 ]	26323.46

Now, the results obtained from table-4 are tested for Pareto-optimality and the following (28) Pareto optimal results are given in Table-5.

Table-5

(pareto optimal results)

Model	V	N	$\bar{t}_1$	$\bar{t}_2$	$[ \bar{F}_L , \bar{F}_R ]$	$\bar{F}_C$
Model-I	0.00258	9	0.65	0.9770380	[ 24947.15 , 29065.76 ]	27006.46
Model-2	0.00488	10	0.5579928	0.9200758	[ 25362.84 , 29687.51 ]	27525.17
Model-3	0.00299	9	0.65	0.7248917	[ 24205.95 , 28440.97 ]	26323.46

ARE YOU SATISFIED WITH THE CURRENT PARETO OPTIMALITY SOLUTION ( OTHER WISE RECHOICE THE MEMBERSHIP FUNCTIONS ) ?

= YES STISFIED

In Table-5, the values of V are quite small and hence, the optimum result in Table-4 are strong Pareto optimum and can be accepted. Still. If the decision-maker / practitioner is not satisfied with the outputs, he / she may perform the above analysis again re-choosing the membership functions or FL , FC and FR , as linear, quadratic and exponential (say). If this second time analysis does not also give the desired result, the DM may perform the analysis with the other possible different combinations ( in this case, 33 times) of the membership functions and can select the most suitable optimum solution for his / her firm for implementation.`

**Result when  $a = 0$  i.e.  $f(t) = bt + c$  :**

**WHEN FUZZY PARAMETER is TFN and  $a = 0$  :**

To illustrate the proposed inventory models, following input data are considered.

**Input Data:**  $C_1=0.4$ ,  $C_2=6.25$ ,  $C_3=420$ ,  $\delta=0.93$ ,  $b=1$ ,  $c=100$ ,  $C_p=15$ ,  $C'=0.15$ ,  $H_1=12$ ,  $H_2=14$ ,  $a_1=0.45$ ,  $a_2=0.65$ ,  $a_3=0.85$  in proper units.

**WHEN FUZZY PARAMETER is TFN :**

Considering the above fuzzy parameter  $\tilde{L}$  as triangular fuzzy number (TFN), the nearest interval approximations according to Grzegorzewski [9] is  $\tilde{L} \equiv [ .55, .75 ]$ .

Following (23) and (25), the problem (21) is solved and the results are presented in the following tables:

**Table-6. Individual Minimum and Maximum of Objective Functions**

Objective functions	Minimum			Maximum		
	Model-I	Model -2	Model -3	Model -I	Model -2	Model -3
$F_L$	23631.27	24034.50	23037.49	23744.09	24147.36	23147.89
$F_R$	27156.41	27552.01	26529.56	27270.01	27665.75	26640.71
$F_C$	25422.14	25821.58	24811.22	25450.64	25850.13	24839.10

Table-7

(Input Data of  $F_L^1, F_L^0, F_R^1, F_R^0, F_C^1, F_C^0$ )

Model	$F_L^1$	$F_L^0$	$F_R^1$	$F_R^0$	$F_C^1$	$F_C^0$
Model-I	23631.27	23740.09	27156.41	27270.00	25422.14	25450.00
Model-2	24034.50	24140.36	27552.01	27660.75	25821.58	25850.00
Model-3	23037.49	23145.89	26529.56	26640.00	24811.22	24838.00

Let, with the above values, the membership functions of the objective functions may be formed of the type as per the Table-9.

Table-8

Objective functions	Type of membership functions
$F_L$	Type-I or Type-II
$F_R$	Type-I or Type-II
$F_C$	Type-I or Type-II

Let, at the beginning, analysis is performed to find optimum  $\lambda$  with the membership function  $F_L, F_C$  as linear (Type-I) and  $F_R$  as Quadratic (Type-II). The optimum value of  $\lambda$  is presented in Table-9.

$\lambda$ - MAXIMUM CALCULATION:

Table-9  
(Optimal value of  $\lambda$ )

Maximum $\lambda$	Model-I	Model-2	Model-3
$\lambda^*$	0.8511807	0.8426335	0.8517809

With this value of  $\lambda^*$ , the objective function  $F_R$  is optimized and the optimum results are:

**Table-10. Optimal Results**

Model	N	$t_1^*$	$t_2^*$	$[ F_L^*, F_R^* ]$	$F_C^*$
Model-I	9	0.65	1.0165930	[ 23647.46 , 27200.23 ]	25423.85
Model-2	10	0.5569648	0.9303536	[ 24051.16 , 27595.15 ]	25823.15
Model-3	9	0.65	0.7722806	[ 23053.56 , 26572.08 ]	24812.82

Now, the results obtained from table-10 are tested for Pareto-optimality and the Pareto optimal results are given in Table-11.

**Table-11. Pareto Optimal Results**

Model	V	N	$\bar{t}_1$	$\bar{t}_2$	$[ \bar{F}_L, \bar{F}_R ]$	$\bar{F}_C$
Model-I	0.00258	9	0.65	1.0165880	[ 23647.46 , 27200.23 ]	25423.85
Model-2	0.00488	10	0.5569666	0.9303436	[ 24051.16 , 27595.15 ]	25823.15
Model-3	0.00299	9	0.65	0.7722778	[ 23053.56 , 26572.08 ]	24812.82

Here again, the values of V are very small and hence the optimal values in Table-10 are strong Pareto-optimum and can be accepted if DM is satisfied. If DM is not satisfied, he / she may perform the analysis with different combinations of membership functions.

## CONCLUSIONS

The present paper proposes a solution procedure for inventory model with time dependent demand rate where demand increases with time and imprecise lead-time. Here, shortages are allowed and backlogged partially. The fuzzy parameter is described by linear / non-linear type membership functions. Fuzzy numbers are then approximated to an interval number. Hence the problem has been converted into a multi-objective inventory problem where the objective functions are represented by left limit, right limit and center of interval function which are to be minimized. To obtain the solution of the deterministic multi-objective inventory problem, the interactive fuzzy solution procedure has been used. The advantage of this procedure is that the decision-maker can easily minimize his worst case. Different scenarios have been considered depending upon the time of placing the order for the next lot. The formulation of the model and the solution procedure presented here are quite general. Here, the results have been presented with imprecise lead-time represented by Triangular Fuzzy Number only. Similarly,

the results can be derived for Parabolic Fuzzy Number and other non-linear fuzzy numbers. Though the problem has been presented in crisp and fuzzy environment, it can be also formulated in fuzzy-stochastic environments.

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**APPENDICES**

$$\begin{aligned}
 \sum_{j=1}^{N+1} T_j^2 &= T_1^2 + T_2^2 + T_3^2 + \dots + T_{N+1}^2 \\
 &= t_3^2 (1^2 + 2^2 + 3^2 + \dots + N^2) \\
 &= \frac{t_3^2}{6} \{ N(N+1)(2N+1) \} \\
 &= \frac{1}{6} \{ N(N+1)(2N+1) \} (t_1+L)^2 \\
 &= \frac{1}{6} \{ N(N+1)(2N+1) \} \cdot \{ t_1 + [L_1, L_2] \}^2 . \\
 &= \frac{1}{6} \{ N(N+1)(2N+1) \} \cdot [t_1+L_1, t_1+L_2] \cdot [t_1+L_1, t_1+L_2] \\
 &= \frac{1}{6} \{ N(N+1)(2N+1) \} \cdot [(t_1+L_1)^2, (t_1+L_2)^2] \\
 &= [ E_1, E_2 ] \\
 \text{where } E_1 &= \frac{1}{6} \{ N(N+1)(2N+1) \} \cdot (t_1+L_1)^2 \\
 \text{and } E_2 &= \frac{1}{6} \{ N(N+1)(2N+1) \} \cdot (t_1+L_2)^2
 \end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^{N+1} T_j &= T_1 + T_2 + T_3 + \dots + T_{N+1} \\
&= t_3 (1 + 2 + 3 + \dots + N) \\
&= \frac{t_3}{2} \{ N(N+1) \} \\
&= \frac{1}{2} \{ N(N+1) \} (t_1 + L) \\
&= \frac{1}{2} \{ N(N+1) \} \cdot \{ t_1 + [L_1, L_2] \} \\
&= \frac{1}{2} \{ N(N+1) \} [t_1 + L_1, t_1 + L_2] \\
&= [G_1, G_2]
\end{aligned}$$

where  $G_1 = \frac{1}{2} \{ N(N+1) \} (t_1 + L_1)$

and  $G_2 = \frac{1}{2} \{ N(N+1) \} (t_1 + L_2)$

$$\begin{aligned}
\sum_{j=1}^{N+1} C_{Hj} &= C_1 \left\{ \frac{a}{2} t_2^2 \sum_{j=1}^{N+1} T_j^2 + \left( \frac{2a}{3} t_2^3 + \frac{b}{2} t_2^2 \right) \sum_{j=1}^{N+1} T_j + \left( \frac{a}{4} t_2^4 + \frac{b}{3} t_2^3 + \frac{c}{2} t_2^2 \right) \right\} \\
&= C_1 \left\{ \frac{a}{2} t_2^2 [E_1, E_2] + \left( \frac{2a}{3} t_2^3 + \frac{b}{2} t_2^2 \right) [G_1, G_2] + \left( \frac{a}{4} t_2^4 + \frac{b}{3} t_2^3 + \frac{c}{2} t_2^2 \right) \right\} \\
&= [M_1, M_2]
\end{aligned}$$

where

$$M_1 = C_1 \left\{ \frac{a}{2} t_2^2 E_1 + \left( \frac{2a}{3} t_2^3 + \frac{b}{2} t_2^2 \right) G_1 + \left( \frac{a}{4} t_2^4 + \frac{b}{3} t_2^3 + \frac{c}{2} t_2^2 \right) \right\}$$

and

where

$$X_1 = \frac{1}{2} \delta C_2 (t_3 - t_2)^2 \left\{ a \cdot E_1 + (2at_2 + b) G_1 + (N+1)(at_2^2 + bt_2 + c) \right\}$$

and

$$X_2 = \frac{1}{2} \delta C_2 (t_3 - t_2)^2 \left\{ a \cdot E_2 + (2at_2 + b) G_2 + (N+1)(at_2^2 + bt_2 + c) \right\}$$

$$\begin{aligned}
\sum_{j=1}^{N+1} R_j &= \delta (t_3 - t_2) \left\{ a \cdot \sum_{j=1}^{N+1} T_j^2 + (2at_2 + b) \sum_{j=1}^{N+1} T_j + (N+1)(at_2^2 + bt_2 + c) \right\} \\
&= \delta (t_3 - t_2) \left\{ a \cdot [E_1, E_2] + (2at_2 + b) [G_1, G_2] + (N+1)(at_2^2 + bt_2 + c) \right\} \\
&= [K_1, K_2]
\end{aligned}$$

where

$$K_1 = \delta (t_3 - t_2) \left\{ a. E_1 + (2at_2 + b) G_1 + (N+1)(at_2^2 + bt_2 + c) \right\}$$

and

$$K_2 = \delta (t_3 - t_2) \left\{ a. E_2 + (2at_2 + b) G_2 + (N+1)(at_2^2 + bt_2 + c) \right\}$$

$$\begin{aligned} F(t_1, t_2) &= \sum_{j=1}^{N+1} C_{Hj} + \sum_{j=1}^{N+1} C_{Sj} + \left( C_p + \frac{C_p'}{L} \right) \sum_{j=1}^{N+1} Q_j + \left( C_p + \frac{C_p'}{L} \right) \sum_{j=1}^{N+1} R_j + \sum_{j=1}^{N+1} C_{3j} \\ &= [M_1, M_2] + [H_1, H_2] + \left[ C_p + \frac{C_p'}{L_2}, C_p + \frac{C_p'}{L_1} \right] [G_1, G_2] + \left[ C_p + \frac{C_p'}{L_2}, C_p + \frac{C_p'}{L_1} \right] \\ &\quad [K_1, K_2] + (N+1) C_3 \end{aligned}$$

$$= [F_L, F_R] \quad \text{where } F_L = M_1 + H_1 + \left( C + \frac{C'}{L_2} \right) G_1 + \left( C + \frac{C'}{L_2} \right) K_1 + (N+1) C_3$$

$$\text{and } F_R = M_2 + H_2 + \left( C + \frac{C'}{L_1} \right) G_2 + \left( C + \frac{C'}{L_1} \right) K_2 + (N+1) C_3$$